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Introduction to the Mathematics of Finance

*From Risk Management
to Options Pricing*

With 55 Figures



Springer

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