PROBABILITY, MARKOV CHAINS, QUEUES, AND SIMULATION

The Mathematical Basis of Performance Modeling

William J. Stewart
## Contents

**Preface and Acknowledgments**

### I PROBABILITY

1 Probability

1.1 Trials, Sample Spaces, and Events

1.2 Probability Axioms and Probability Space

1.3 Conditional Probability

1.4 Independent Events

1.5 Law of Total Probability

1.6 Bayes' Rule

1.7 Exercises

2 Combinatorics—The Art of Counting

2.1 Permutations

2.2 Permutations with Replacements

2.3 Permutations without Replacement

2.4 Combinations without Replacement

2.5 Combinations with Replacements

2.6 Bernoulli (Independent) Trials

2.7 Exercises

3 Random Variables and Distribution Functions

3.1 Discrete and Continuous Random Variables

3.2 The Probability Mass Function for a Discrete Random Variable

3.3 The Cumulative Distribution Function

3.4 The Probability Density Function for a Continuous Random Variable

3.5 Functions of a Random Variable

3.6 Conditioned Random Variables

3.7 Exercises

4 Joint and Conditional Distributions

4.1 Joint Distributions

4.2 Joint Cumulative Distribution Functions

4.3 Joint Probability Mass Functions

4.4 Joint Probability Density Functions

4.5 Conditional Distributions

4.6 Convolutions and the Sum of Two Random Variables

4.7 Exercises

5 Expectations and More

5.1 Definitions

5.2 Expectation of Functions and Joint Random Variables

5.3 Probability Generating Functions for Discrete Random Variables
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4</td>
<td>Moment Generating Functions</td>
<td>103</td>
</tr>
<tr>
<td>5.5</td>
<td>Maxima and Minima of Independent Random Variables</td>
<td>108</td>
</tr>
<tr>
<td>5.6</td>
<td>Exercises</td>
<td>110</td>
</tr>
<tr>
<td>6</td>
<td><strong>Discrete Distribution Functions</strong></td>
<td>115</td>
</tr>
<tr>
<td>6.1</td>
<td>The Discrete Uniform Distribution</td>
<td>115</td>
</tr>
<tr>
<td>6.2</td>
<td>The Bernoulli Distribution</td>
<td>116</td>
</tr>
<tr>
<td>6.3</td>
<td>The Binomial Distribution</td>
<td>117</td>
</tr>
<tr>
<td>6.4</td>
<td>Geometric and Negative Binomial Distributions</td>
<td>120</td>
</tr>
<tr>
<td>6.5</td>
<td>The Poisson Distribution</td>
<td>124</td>
</tr>
<tr>
<td>6.6</td>
<td>The Hypergeometric Distribution</td>
<td>127</td>
</tr>
<tr>
<td>6.7</td>
<td>The Multinomial Distribution</td>
<td>128</td>
</tr>
<tr>
<td>6.8</td>
<td>Exercises</td>
<td>130</td>
</tr>
<tr>
<td>7</td>
<td><strong>Continuous Distribution Functions</strong></td>
<td>134</td>
</tr>
<tr>
<td>7.1</td>
<td>The Uniform Distribution</td>
<td>134</td>
</tr>
<tr>
<td>7.2</td>
<td>The Exponential Distribution</td>
<td>136</td>
</tr>
<tr>
<td>7.3</td>
<td>The Normal or Gaussian Distribution</td>
<td>141</td>
</tr>
<tr>
<td>7.4</td>
<td>The Gamma Distribution</td>
<td>145</td>
</tr>
<tr>
<td>7.5</td>
<td>Reliability Modeling and the Weibull Distribution</td>
<td>149</td>
</tr>
<tr>
<td>7.6</td>
<td>Phase-Type Distributions</td>
<td>155</td>
</tr>
<tr>
<td>7.6.1</td>
<td>The Erlang-2 Distribution</td>
<td>155</td>
</tr>
<tr>
<td>7.6.2</td>
<td>The Erlang-r Distribution</td>
<td>158</td>
</tr>
<tr>
<td>7.6.3</td>
<td>The Hypoexponential Distribution</td>
<td>162</td>
</tr>
<tr>
<td>7.6.4</td>
<td>The Hyperexponential Distribution</td>
<td>164</td>
</tr>
<tr>
<td>7.6.5</td>
<td>The Coxian Distribution</td>
<td>166</td>
</tr>
<tr>
<td>7.6.6</td>
<td>General Phase-Type Distributions</td>
<td>168</td>
</tr>
<tr>
<td>7.6.7</td>
<td>Fitting Phase-Type Distributions to Means and Variances</td>
<td>171</td>
</tr>
<tr>
<td>7.7</td>
<td>Exercises</td>
<td>176</td>
</tr>
<tr>
<td>8</td>
<td><strong>Bounds and Limit Theorems</strong></td>
<td>180</td>
</tr>
<tr>
<td>8.1</td>
<td>The Markov Inequality</td>
<td>180</td>
</tr>
<tr>
<td>8.2</td>
<td>The Chebychev Inequality</td>
<td>181</td>
</tr>
<tr>
<td>8.3</td>
<td>The Chemoff Bound</td>
<td>182</td>
</tr>
<tr>
<td>8.4</td>
<td>The Laws of Large Numbers</td>
<td>182</td>
</tr>
<tr>
<td>8.5</td>
<td>The Central Limit Theorem</td>
<td>184</td>
</tr>
<tr>
<td>8.6</td>
<td>Exercises</td>
<td>187</td>
</tr>
</tbody>
</table>

**II MARKOV CHAINS**

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td><strong>Discrete- and Continuous-Time Markov Chains</strong></td>
<td>193</td>
</tr>
<tr>
<td>9.1</td>
<td>Stochastic Processes and Markov Chains</td>
<td>193</td>
</tr>
<tr>
<td>9.2</td>
<td>Discrete-Time Markov Chains: Definitions</td>
<td>195</td>
</tr>
<tr>
<td>9.3</td>
<td>The Chapman-Kolmogorov Equations</td>
<td>202</td>
</tr>
<tr>
<td>9.4</td>
<td>Classification of States</td>
<td>206</td>
</tr>
<tr>
<td>9.5</td>
<td>Irreducibility</td>
<td>214</td>
</tr>
<tr>
<td>9.6</td>
<td>The Potential, Fundamental, and Reachability Matrices</td>
<td>218</td>
</tr>
<tr>
<td>9.6.1</td>
<td>Potential and Fundamental Matrices and Mean Time to Absorption</td>
<td>219</td>
</tr>
<tr>
<td>9.6.2</td>
<td>The Reachability Matrix and Absorption Probabilities</td>
<td>223</td>
</tr>
</tbody>
</table>
## Contents

11.2 Birth-Death Processes: The $M/M\lambda$ Queue 402  
11.2.1 Description and Steady-State Solution 402  
11.2.2 Performance Measures 406  
11.2.3 Transient Behavior 412  
11.3 General Birth-Death Processes 413  
11.3.1 Derivation of the State Equations 413  
11.3.2 Steady-State Solution 415  
11.4 Multiserver Systems 419  
11.4.1 The $M/M/c$ Queue 419  
11.4.2 The $M/M/\infty$ Queue 425  
11.5 Finite-Capacity Systems—The $M/M/K$ Queue 425  
11.6 Multiserver, Finite-Capacity Systems—The $M/M/c/K$ Queue 432  
11.7 Finite-Source Systems—The $M/M/cf/M$ Queue 434  
11.8 State-Dependent Service 437  
11.9 Exercises 438  

12 Queues with Phase-Type Laws: $\{euts\}$ Matrix-Geometric Method 444  
12.1 The Erlang-r Service Model—The $M/E\lambda$ Queue 444  
12.2 The Erlang-r Arrival Model—The $E_r/M/1$ Queue 450  
12.3 The $M/Ho/JX$ and $H?/M/\lambda$ Queues 454  
12.4 Automating the Analysis of Single-Server Phase-Type Queues 458  
12.5 The $H/T/E_\lambda$ Queue and General $Ph/Ph\lambda$ Queues 460  
12.6 Stability Results for $Ph/Ph\lambda$ Queues 466  
12.7 Performance Measures for $Ph/Ph/1$ Queues 468  
12.8 Matlab code for $Pli/Ph\lambda$ Queues 469  
12.9 Exercises 471  

13 The z-Transform Approach to Solving Markovian Queues 475  
13.1 The $z$-Transform 475  
13.2 The Inversion Process 478  
13.3 Solving Markovian Queues using $z$-Transforms 484  
13.3.1 The $z$-Transform Procedure 484  
13.3.2 The $M/M\lambda$ Queue Solved using $z$-Transforms 484  
13.3.3 The $M/M\lambda$ Queue with Arrivals in Pairs 486  
13.3.4 The $M/E\lambda$ Queue Solved using $z$-Transforms 488  
13.3.5 The $E_r/M\lambda$ Queue Solved using $z$-Transforms 496  
13.3.6 Bulk Queueing Systems 503  
13.4 Exercises 506  

14 The $M/G/l$ and $G/M/l$ Queues 509  
14.1 Introduction to the $M/G/\lambda$ Queue 509  
14.2 Solution via an Embedded Markov Chain 510  
14.3 Performance Measures for the $M/G/\lambda$ Queue 515  
14.3.1 The Pollaczek-Khintchine Mean Value Formula 515  
14.3.2 The Pollaczek-Khintchine Transform Equations 518  
14.4 The $M/G/\lambda$ Residual Time: Remaining Service Time 523  
14.5 The $A/G/1$ Busy Period 526  
14.6 Priority Scheduling 531  
14.6.1 $M/M\lambda$: Priority Queue with Two Customer Classes 531  
14.6.2 $M/G/l$: Nonpreemptive Priority Scheduling 533
14.6.3 M/G/l: Preempt-Resume Priority Scheduling 536
14.6.4 A Conservation Law and SPTF Scheduling 538
14.7 The M/G/l/K Queue 542
14.8 The G/M/l Queue 546
14.9 The G/M/l/K Queue 551
14.10 Exercises 553

15 Queueing Networks 559
15.1 Introduction 559
  15.1.1 Basic Definitions 559
  15.1.2 The Departure Process—Burke's Theorem 560
  15.1.3 Two M/M/l Queues in Tandem 562
15.2 Open Queueing Networks 563
  15.2.1 Feedforward Networks 563
  15.2.2 Jackson Networks 563
  15.2.3 Performance Measures for Jackson Networks 567
15.3 Closed Queueing Networks 568
  15.3.1 Definitions 568
  15.3.2 Computation of the Normalization Constant: Buzen's Algorithm 570
  15.3.3 Performance Measures 577
15.4 Mean Value Analysis for Closed Queueing Networks 582
15.5 The Flow-Equivalent Server Method 591
15.6 Multiclass Queueing Networks and the BCMP Theorem 594
  15.6.1 Product-Form Queueing Networks 595
  15.6.2 The BCMP Theorem for Open, Closed, and Mixed Queueing Networks 598
15.7 Java Code 602
15.8 Exercises 607

IV SIMULATION 611

16 Some Probabilistic and Deterministic Applications of Random Numbers 613
16.1 Simulating Basic Probability Scenarios 613
16.2 Simulating Conditional Probabilities, Means, and Variances 618
16.3 The Computation of Definite Integrals 620
16.4 Exercises 623

17 Uniformly Distributed "Random" Numbers 625
17.1 Linear Recurrence Methods 626
17.2 Validating Sequences of Random Numbers 630
  17.2.1 The Chi-Square "Goodness-of-Fit" Test 630
  17.2.2 The Kolmogorov-Smirnov Test 633
  17.2.3 "Run" Tests 634
  17.2.4 The "Gap" Test 640
  17.2.5 The "Poker" Test 641
  17.2.6 Statistical Test Suites 644
17.3 Exercises 644
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>B.3</td>
<td>Vector and Matrix Norms</td>
<td>723</td>
</tr>
<tr>
<td>B.4</td>
<td>Vector Spaces</td>
<td>724</td>
</tr>
<tr>
<td>B.5</td>
<td>Determinants</td>
<td>726</td>
</tr>
<tr>
<td>B.6</td>
<td>Systems of Linear Equations</td>
<td>728</td>
</tr>
<tr>
<td>B.6.1</td>
<td>Gaussian Elimination and EU Decompositions</td>
<td>730</td>
</tr>
<tr>
<td>B.7</td>
<td>Eigenvalues and Eigenvectors</td>
<td>734</td>
</tr>
<tr>
<td>B.8</td>
<td>Eigenproperties of Decomposables, Nearly Decomposables, and Cyclic</td>
<td>738</td>
</tr>
<tr>
<td></td>
<td>Stochastic Matrices</td>
<td></td>
</tr>
<tr>
<td>B.8.1</td>
<td>Normal Form</td>
<td>738</td>
</tr>
<tr>
<td>B.8.2</td>
<td>Eigenvalues of Decomposable Stochastic Matrices</td>
<td>739</td>
</tr>
<tr>
<td>B.8.3</td>
<td>Eigenvectors of Decomposable Stochastic Matrices</td>
<td>741</td>
</tr>
<tr>
<td>B.8.4</td>
<td>Nearly Decomposable Stochastic Matrices</td>
<td>743</td>
</tr>
<tr>
<td>B.8.5</td>
<td>Cyclic Stochastic Matrices</td>
<td>744</td>
</tr>
<tr>
<td></td>
<td><strong>Bibliography</strong></td>
<td>745</td>
</tr>
<tr>
<td></td>
<td><strong>Index</strong></td>
<td>749</td>
</tr>
</tbody>
</table>