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# The Elements of Statistical Learning

Data Mining, Inference, and Prediction

Second Edition

**inger**

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