

# The Calculus of Retirement Income

*Financial Models for Pension Annuities  
and Life Insurance*

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# Contents

*List of Figures and Tables*

*page x*

## I MODELS OF ACTUARIAL FINANCE

1	Introduction and Motivation	3
1.1	The Drunk Gambler Problem	3
1.2	The Demographic Picture	5
1.3	The Ideal Audience	9
1.4	Learning Objectives	10
1.5	Acknowledgments	12
1.6	Appendix: Drunk Gambler Solution	14
2	Modeling the Human Life Cycle	17
2.1	The Next Sixty Years of Your Life	17
2.2	Future Value of Savings	18
2.3	Present Value of Consumption	20
2.4	Exchange Rate between Savings and Consumption	22
2.5	A Neutral Replacement Rate	26
2.6	Discounted Value of a Life-Cycle Plan	27
2.7	Real vs. Nominal Planning with Inflation	28
2.8	Changing Investment Rates over Time	30
2.9	Further Reading	32
2.10	Problems	33
3	Models of Human Mortality	34
3.1	Mortality Tables and Rates	34
3.2	Conditional Probability of Survival	35
3.3	, Remaining Lifetime Random Variable	37
3.4	Instantaneous Force of Mortality	38
3.5	The ODE Relationship	39
3.6	Moments in Your Life	41

3.7	Median vs. Expected Remaining Lifetime	44
3.8	Exponential Law of Mortality	45
3.9	Gompertz-Makeham Law of Mortality	46
3.10	Fitting Discrete Tables to Continuous Laws	49
3.11	General Hazard Rates	51
3.12	Modeling Joint Lifetimes	53
3.13	Period vs. Cohort Tables	55
3.14	Further Reading	59
3.15	Notation	60
3.16	Problems	60
3.17	Technical Note: Incomplete Gamma Function in Excel	61
3.18	Appendix: Normal Distribution and Calculus Refresher	62
4	Valuation Models of Deterministic Interest	64
4.1	Continuously Compounded Interest Rates?	64
4.2	Discount Factors	66
4.3	How Accurate Is the Rule of 72?	67
4.4	Zero Bonds and Coupon Bonds	68
4.5	Arbitrage: Linking Value and Market Price	70
4.6	Term Structure of Interest Rates	72
4.7	Bonds: Nonflat Term Structure	73
4.8	Bonds: Nonconstant Coupons	74
4.9	Taylor's Approximation	75
4.10	Explicit Values for Duration and Convexity	76
4.11	Numerical Examples of Duration and Convexity	78
4.12	Another Look at Duration and Convexity	80
4.13	Further Reading	81
4.14	Notation	82
4.15	Problems	82
5	Models of Risky Financial Investments	83
5.1	Recent Stock Market History	83
5.2	Arithmetic Average Return versus Geometric Average Return	86
5.3	A Long-Term Model for Risk	88
5.4	Introducing Brownian Motion	91
5.5	Index Averages and Index Medians	97
5.6	The Probability of Regret	98
5.7	Focusing on the Rate of Change	100
5.8	How to Simulate a Diffusion Process	101
5.9	Asset Allocation and Portfolio Construction	102
5.10	Space-Time Diversification	104
5.11	Further Reading	107
5.12	Notation	108
5.13	Problems	108

6	Models of Pension Life Annuities	110
6.1	Motivation and Agenda	110
6.2	Market Prices of Pension Annuities	110
6.3	Valuation of Pension Annuities: General	114
6.4	Valuation of Pension Annuities: Exponential	115
6.5	The Wrong Way to Value Pension Annuities	115
6.6	Valuation of Pension Annuities: Gompertz-Makeham	116
6.7	How Is the Annuity's Income Taxed?	119
6.8	Deferred Annuities: Variation on a Theme	121
6.9	Period Certain versus Term Certain	123
6.10	Valuation of Joint and Survivor Pension Annuities	125
6.11	Duration of a Pension Annuity	128
6.12	Variable vs. Fixed Pension Annuities	130
6.13	Further Reading	134
6.14	Notation	136
6.15	Problems	136
7	Models of Life Insurance	138
7.1	A Free (Last) Supper?	138
7.2	Market Prices of Life Insurance	138
7.3	The Impact of Health Status	139
7.4	How Much Life Insurance Do You Need?	140
7.5	Other Kinds of Life Insurance	142
7.6	Value of Life Insurance: Net Single Premium	143
7.7	Valuing Life Insurance Using Pension Annuities	145
7.8	Arbitrage Relationship	147
7.9	Tax Arbitrage Relationship	148
7.10	Value of Life Insurance: Exponential Mortality	149
7.11	Value of Life Insurance: GoMa Mortality	149
7.12	Life Insurance Paid by Installments	150
7.13	NSP: Delayed and Term Insurance	150
7.14	Variations on Life Insurance	151
7.15	What If You Stop Paying Premiums?	154
7.16	Duration of Life Insurance	157
7.17	Following a Group of Policies	159
7.18	The Next Generation: Universal Life Insurance	160
7.19	Further Reading	162
7.20	Notation	162
7.21	Problems	162
8	Models of DB vs. DC Pensions	164
8.1	A Choice of Pension Plans	164
8.2	The Core of Defined Contribution Pensions	165
8.3	The Core of Defined Benefit Pensions	169

8.4	What Is the Value of a DB Pension Promise?	172
8.5	Pension Funding and Accounting	176
8.6	Further Reading	180
8.7	Notation	181
8.8	Problems	182
II WEALTH MANAGEMENT: APPLICATIONS AND IMPLICATIONS		
9	Sustainable Spending at Retirement	185
9.1	Living in Retirement	185
9.2	Stochastic Present Value	187
9.3	Analytic Formula: Sustainable Retirement Income	190
9.4	The Main Result: Exponential Reciprocal Gamma	192
9.5	Case Study and Numerical Examples	193
9.6	Increased Sustainable Spending <i>without</i> More Risk?	202
9.7	Conclusion	206
9.8	Further Reading	208
9.9	Problems	208
9.10	Appendix: Derivation of the Formula	209
10	Longevity Insurance Revisited	215
10.1	To Annuitize or Not To Annuitize?	215
10.2	Five 95-Year-Olds Playing Bridge	216
10.3	The Algebra of Fixed and Variable Tontines	218
10.4	Asset Allocation with Tontines	220
10.5	A First Look at Self-Annuitization	225
10.6	The Implied Longevity Yield	226
10.7	Advanced-Life Delayed Annuities	234
10.8	Who Incurs Mortality Risk and Investment Rate Risk?	241
10.9	Further Reading	244
10.10	Notation	245
10.11	Problems	245
III ADVANCED TOPICS		
11	Options within Variable Annuities	249
11.1	To Live and Die in VA	249
11.2	The Value of Paying by Installments	252
11.3	A Simple Guaranteed Minimum Accumulation Benefit	257
11.4	The Guaranteed Minimum Death Benefit	258
11.5	Special Case: Exponential Mortality	259
11.6	The Guaranteed Minimum Withdrawal Benefit	262
11.7	Further Reading	268
11.8	Notation	269

12	The Utility of Annuitization	270
12.1	What Is the Protection Worth?	270
12.2	Models of Utility, Value, and Price	271
12.3	The Utility Function and Insurance	272
12.4	Utility of Consumption and Lifetime Uncertainty	274
12.5	Utility and Annuity Asset Allocation	278
12.6	The Optimal Timing of Annuitization	281
12.7	The Real Option to Defer Annuitization	282
12.8	Advanced RODA Model	287
12.9	Subjective vs. Objective Mortality	289
12.10	Variable vs. Fixed Payout Annuities	290
12.11	Further Reading	291
12.12	Notation	292
13	Final Words	293
14	Appendix	295
	<i>Bibliography</i>	301
	<i>Index</i>	309